The Overlapping-Generations Model

I. Introduction

The beautiful and mysterious overlapping-generation model goes back to Samuelson (1958), or possibly to Allais (1947), if you can read French, and it has kept generations of mathematically minded economists happily engaged.

II. A Very Simple Model with Chocolate Bars

It was a dark and stormy night.[1] (One does not usually get to begin an economics lecture with such lurid prose.) A traveller arrives at an inn. The innkeeper informs the traveller that he has an infinite number of beds, but unfortunately they are all taken. There is a solution, however. Each guest can simply move down a bed. Karl Shell (1971) tells this story from George Gamow's 1947 book *One Two Three … Infinity* to suggest the intuition behind the famous welfare result in the overlapping generations model.

Suppose that time occurs in discrete intervals; the model begins in period zero and stretches to infinity. Each period a single two-period-lived consumer is born. This produces the key demographic feature of the model. A consumer born at time t can trade with the consumer born in period $t - 1$ in period t and the consumer born in period $t + 1$ in period $t + 1$ and no one else.

The model has a single good—chocolate bars—and every consumer receives one at birth. A limiting feature of the chocolate bars is that they cannot be stored. Perhaps the climate is too warm and they melt. Each consumer wishes solely to consume as much chocolate as they can and has no concern over when the chocolate is eaten. Thus, a consumer's utility can be represented by the total amount of chocolate that they consume.

[1] This simple model is appropriate for an undergraduate lecture and can be well received, especially if you bring a basket of chocolate bars.

Clearly, in equilibrium, each consumer eats his or her own chocolate bar. Let p_t be the time-t price of a chocolate bar. Then this equilibrium can be supported by any price sequence where the price of chocolate rises (weakly) over time.

The First Theorem of Welfare Economics says that any competitive allocation is Pareto optimal. This, however, is an example of an economy where this theorem fails. To see this, suppose that each young person gives the old person who is alive at the same time their chocolate bar. Clearly, no one is worse off. Everyone born after the model begins consumes the same amount of chocolate as they would without the transfer. The only change is that they eat their chocolate when they are old rather than when they are young. The old consumer alive in period zero—born before the model begins—is strictly better off. With the transfer this consumer receives a chocolate bar that they would not otherwise have had.

How can this Pareto improvement be attained in a decentralized way? Suppose that the old consumer alive in period one had received a chocolate bar that was different from any of the ones received by those born after the model begins: it had a wrapper. Suppose that, after eating his chocolate bar, this consumer kept the wrapper and in period 1 they presented the wrapper to the young consumer then alive as a claim on a chocolate bar. If indeed it is a common belief that a wrapper is a claim on a chocolate bar (that is, everyone believes it; everyone believes that everyone else believes it; everyone believes that everyone else believes that everyone else believes it, ad infinitum) then the young consumer is willing to trade their chocolate bar for the wrapper and so is every young consumer ever after. This affects the Pareto-improving transfer.

What causes the First Theorem of Welfare Economics to fail? Perhaps most economists when initially confronted with this question would say that it is because not everyone can trade with everyone else. But, this is not the problem. It does not matter that the person born in, say, period three cannot trade with the person born in, say, period ten. Neither has anything the other wants. A chocolate bar received in period three has long gone bad by period ten and a chocolate bar received in period ten has no benefit to a person dead by the end of period four.

To see what is going wrong, consider the proof of the First Theorem of Welfare Economics in a simple endowment economy. The strategy is show that it is impossible to find a feasible allocation that is Pareto superior to the competitive equilibrium. This is done by first noting that since a Pareto superior allocation would merely be a rearrangement of the endowment it cannot cost more (at competitive equilibrium prices) than the endowment. Second, if there is non-satiation and everyone satisfies their budget constraint than the cost of the competitive equilibrium must equal the cost of the endowment. Third, if a person is indifferent between their consumption basket in a Pareto superior allocation and their competitive equilibrium consumption basket then the Pareto superior allocation must cost at least as much or they could have bought a consumption basket that they strictly preferred to the Pareto superior allocation consumption basket and thus to their equilibrium basket. If a person strictly prefers their Pareto superior allocation consumption basket, then it must cost strictly more than their competitive equilibrium consumption basket or they would have bought it. Thus, a Pareto superior consumption basket must cost strictly more than the competitive allocation consumption basket. But, this is not consistent with steps one and two.

More formally, imagine an economy with a finite number *I* of consumers indexed by $i = 1, \ldots, I$ and a finite number *K* of goods indexed by $k = 1, \ldots, K$. Each consumer *i* is endowed with $e_i^k > 0$ units of the good k. Denote the competitive equilibrium price of good k by p_k^i and consumer i's competitive equilibrium consumption of good k by c_k^i. Suppose that there exists a Pareto superior allocation and let a_k^i be consumer i's consumption of good k in this allocation. The Pareto superior allocation must be feasible, hence, $\sum_{i=1}^{I} a_k^i \le e_k \equiv \sum_{i=1}^{I} e_k$. This implies

$$\sum_{k=1}^{K} p_k \sum_{i=1}^{I} a_k^i \le \sum_{k=1}^{K} p_k e_k. \tag{1}$$

By non-satiation and Walras' Law[2]

$$\sum_{1=1}^{I} \sum_{k=1}^{K} p_k c_k^i = \sum_{k=1}^{K} p_k e_k. \tag{2}$$

Each consumer must like their Pareto superior allocation at least as much as their competitive equilibrium allocation and at least one consumer strictly prefers their Pareto superior consumption basket. Suppose that a consumer i is indifferent between their Pareto superior allocation and their competitive equilibrium allocation. Than the Pareto superior allocation cannot cost less or they could have bought an allocation that they strictly preferred to their Pareto superior allocation and hence to their equilibrium allocation. Thus, $\sum_{k=1}^{K} p_k c_k^i \le \sum_{k=1}^{K} p_k a_k^i$. Suppose that a consumer i strictly prefers their Pareto superior allocation. Then, it must cost strictly more or they would have bought it. Thus, for this i, $\sum_{k=1}^{K} p_k c_k^i < \sum_{k=1}^{K} p_k a_k^i$. Summing over all consumers

[2] Non-satiation implied everyone satisfies their budget constraints exactly and hence we have Walras Law: the value of excess demand at equilibrium prices is zero.

$$\sum_{i=1}^{I}\sum_{k=1}^{K}p_k c_k^i < \sum_{i=1}^{I}\sum_{k=1}^{K}p_k a_k^i. \tag{3}$$

By equations (1) to (3)

$$\sum_{k=1}^{K}p_k\sum_{i=1}^{I}a_k^i \le \sum_{k=1}^{K}p_k e_k = \sum_{1=1}^{I}\sum_{k=1}^{K}p_k c_k^i < \sum_{i=1}^{I}\sum_{k=1}^{K}p_k a_k^i = \sum_{k=1}^{K}p_k\sum_{i=1}^{I}a_k^i. \tag{4}$$

But, this is a contradiction.

This proof does not work for the overlapping generations model because there are an infinite number of consumers and an infinite number of goods: period 1 chocolate bars, period 2 chocolate bars, and so on. Hence it is not legitimate to swap the order of summation (Fubini's Theorem) in equation (1.4). The problem in the overlapping generations model is that the value of endowments at equilibrium prices might not be finite and thus consuming more of the good might not violate a resource constraint. This is analogous to what is happening in the innkeeper's problem.

III. A Model with Fiat Money

This section considers Wallace's (1980) model with fiat money. Suppose that in each period N_t two-period lived consumers are born, where $N_{t+1} / N_t = n > 0$. While it is typical to assume that consumers live for two periods, it is possible to assume that they live for more periods; that they die with positive probability each period (Blanchard (1985)) or that they never die (Weill (1989)). The important thing is that there are always new arrivals.

Each consumer receives a constant endowment y of the single good. It is not important that there is a single good or that the endowment is constant. The good can be consumed or it can be stored. If one unit of the good is stored, than x units of the good are received in the next period. The consumer born in period t receives a monetary transfer with a real value of T_{t+1} when old. Denote this consumer's consumption when young by c_t^y, their consumption when old by c_{t+1}^o, the amount they store by k_t and the amount of money they save by M_t. Let p_t be the price of the good at time t.

The consumer born at time t has preferences represented by $U(c_t^y, c_{t+1}^o)$, where the utility function U has the usual nice properties. In, U is homothetic.[3] This is not key, but it simplifies the notation. The problem of the consumer is to maximize the budget constraint subject to the budget constraints

$$\begin{aligned} c_t^y &= y - k_t - M_t / P_t \\ c_{t+1}^o &= xk_t + M_t / P_t + T_{t+1}. \end{aligned} \tag{5}$$

There will always be an equilibrium without valued fiat money and in this case consumer optimization implies

$$U_1(y - k_t, xk_t) / U_2(y - k_t, xk) = x. \tag{6}$$

By homotheticity, we can write the left-hand side in equation (6) as $v((y - k_t) / xk_t)$, where v is the strictly decreasing marginal rate of substitution function. Suppose that $y = f(x)$, where f is a strictly monotonic function. Let the inverse function of f be denoted by f^{-1}. Then $f^{-1}(f(x)) = x$. Differentiating both sides with respect to x we have $f^{-1}{}'(f(x))f'(x) = f^{-1}{}'(y)f'(x) = 1 \Rightarrow f^{-1}{}'(y) = 1/ f'(x)$. Thus, v has an inverse function and its derivative is equal to the reciprocal of the v. Denote this inverse function by h. Then by equation (6),

3 A homothetic utility function is one that can be expressed as a strictly positive monotonic function of a homogeneous utility function. It has property that the marginal rates of substitution are homogeneous of degree one. Geometrically, the slopes of the indifference curves are constant on any ray through the origin.

$$y - k_t / xk_t = h(x). \tag{7}$$

Thus, the non-monetary equilibrium has constant storage equal to y/[1 + xh(x)]. If there is an equilibrium with valued fiat money than the first-order condition for the consumer's problem is

$$\begin{aligned} &\frac{y - M_t / P_t}{M_t / P_{t+1} + T_{t+1}} = h\left(\frac{P_t}{P_{t+1}}\right) \text{ and } k_t = 0 \text{ if } \frac{P_t}{P_{t+1}} > x \\ &\frac{y - k_t - M_t / P_t}{M_t / P_{t+1} + xk_{t+1} + T_{t+1}} = h\left(\frac{P_t}{P_{t+1}}\right) \text{ if } \frac{P_t}{P_{t+1}} = x. \end{aligned} \tag{8}$$

Market clearing requires $N_t M_t = M_t^s$. Substituting this into equations (8) yields

$$\begin{aligned} &\frac{y - M_t^s / \left(N_t P_t\right)}{M_t^s / \left(N_t P_{t+1}\right) + T_{t+1}} = h\left(\frac{P_t}{P_{t+1}}\right) \text{ and } k_t = 0 \text{ if } \frac{P_t}{P_{t+1}} > x \\ &\frac{y - k_t - M_t^s / \left(N_t P_t\right)}{M_t^s / \left(N_t P_{t+1}\right) + xk_{t+1} + T_{t+1}} = h\left(\frac{P_t}{P_{t+1}}\right) \text{ if } \frac{P_t}{P_{t+1}} = x. \end{aligned} \tag{9}$$

The money supply increases at a strictly positive constant rate $z = M_{t+1}^s / M_t^s$. The government transfers this increase to the old so that $T_{t+1} = (M_{t+1}^s - M_t^s) / (N_t P_t) = [(z-1) M_t^s] / (N_t P_t)$. Substitute this into equation (9).

$$\begin{aligned} &\frac{y - M_t^s / \left(N_t P_t\right)}{M_{t+1}^s / \left(N_t P_{t+1}\right)} = h\left(\frac{P_t}{P_{t+1}}\right) \text{ and } k_t = 0 \text{ if } \frac{P_t}{P_{t+1}} > x \\ &\frac{y - k_t - M_t^s / \left(N_t P_t\right)}{M_{t+1}^s / \left(N_t P_{t+1}\right) + xk_{t+1}} = h\left(\frac{P_t}{P_{t+1}}\right) \text{ if } \frac{P_t}{P_{t+1}} = x. \end{aligned} \tag{10}$$

Let $m_t \equiv M_t^s / \left(N_t P_t\right)$. Substitute this into equation (10)

$$\begin{aligned} &\frac{y - m_t}{nm_{t+1}} = h\left(\frac{nm_{t+1}}{zm_t}\right) \text{and } k_t = 0 \text{ if } \frac{m_{t+1}}{m_t} > \frac{zx}{n} \\ &\frac{y - k_t - m_t}{nm_{t+1} + xk_{t+1}} = h\left(\frac{nm_{t+1}}{zm_t}\right) \text{ if } \frac{m_{t+1}}{m_t} = \frac{zx}{n}. \end{aligned} \tag{11}$$

Clearly $xz/n \leq 1$ is necessary for an equilibrium with valued money. Otherwise $m_{t+1}/m_t = zx/n > 1$ and real balances go to infinity, which is inconsistent with their being less than y. If $xz/n \leq 1$ than a monetary equilibrium exists. Suppose real balances are constant. Then by the first equation in (11) $(y-m)/(nm) = h(n/z)$. The left-hand side of equation (11) is strictly decreasing in m. As m goes to y it goes to zero; as m goes to zero it goes to infinity. The right-hand side is a constant. Thus, a stationary equilibrium exists. There may exist many other non-stationary equilibria without storage and stationary equilibria with constant storage is possible.

If $x > n$, then a non-monetary equilibrium or an equilibrium with strictly positive stationary storage is non-optimal. To see this, note that such an equilibrium has $N_t c^y + N_{t-1} c^o + N_t k = N_t y + N_{t-1} xk$. Thus, it has $nc^y + c^o = ny + (x-n)k$. Instead of this equilibrium, let k = 0 and have each of the young give k to the old. Then the old get nk from the young instead of xk from their storage. Everyone is at least as well off (and better off if $n > x$) and the old who were born in period zero are are strictly better off because they got a transfer where they did not before.

Other results are possible. If $x > n$ then any equilibrium is optimal. This can be proved by showing that there is no sequence of transfers from the young to the old that is Pareto improving. It can also be shown that if xz/n is less than or equal to one, then the stationary monetary equilibrium is optimal if and only if z is less than or equal to one.

IV. A Model with Fiat Money

This section considers a slightly simplified version of Kareken and Wallace's (1981) model of exchange rates. The model is the same as the one in the previous section except that there are two countries, each with its own fiat money, a single consumer is born in period in each country and storage is not possible. The countries are referred to as the home country (country h) and the foreign country (country f). Preferences are constant across countries and the money supplies grow at the same constant rate z.

Suppose initially that consumers only hold their own country's currency. Let p_t be the price of the good in terms of home money and let e_t be the price of foreign money in terms of the home money. Let y^i be the endowment in country *i* and let T_t^i be the real value of the time-t monetary transfer. Let c_t^{yi} and c_{t+1}^{oi} be the time-*t* and time-*t*+1 consumption of the consumer born at time *t* in country *i* and let M_t^i be their demand for money *i*, *i* = *h*,*f*. Then the budget constraints of the person born at time t in the home country are

$$\begin{aligned} c_t^{yh} &= y^h - M_t^h / P_t \\ c_{t+1}^{oh} &= M_t^h / P_{t+1} + T_{t+1}^h. \end{aligned} \tag{12}$$

The budget constraints of the person born at time t in the foreign country are

$$\begin{aligned} c_t^{yf} &= y^f - e_t M_t^f / P_t \\ c_{t+1}^{of} &= e_{t+1} M_t^f / P_{t+1} + T_{t+1}^f. \end{aligned} \tag{13}$$

The first-order conditions are

$$\begin{aligned} \frac{y^h - M_t^h / P_t}{M_t^h / P_{t+1} + T_{t+1}^h} &= h\left(\frac{P_t}{P_{t+1}}\right) \\ \frac{y^f - e_t M_t^f / P_t}{e_{t+1} M_t^f / P_{t+1} + T_{t+1}^f} &= h\left(\frac{e_{t+1} P_t}{e_t P_{t+1}}\right). \end{aligned} \tag{14}$$

Market clearing requires $M_t^i = M_t^{si}, i = h, f.$ Transfers satisfy $T_{t+1}^h = (z-1) M_t^{sh} / P_t$ and $T_{t+1}^f = (z-1) e_t M_t^{sf} / P_t$. Substituting this into (14) yields

$$\begin{aligned} \frac{y^h - M_t^h / P_t}{M_{t+1}^h / P_{t+1}} &= h\left(\frac{P_t}{P_{t+1}}\right) \\ \frac{y^f - e_t M_t^f / P_t}{e_{t+1} M_{t+1}^f / P_{t+1}} &= h\left(\frac{e_{t+1} P_t}{e_t P_{t+1}}\right). \end{aligned} \tag{15}$$

Let $m_t^h \equiv M_t^{sh} / P_t$ and $m_t^f \equiv e_t M_t^{sf} / P_t$. Substituting this into equation (15) yields

$$\frac{y^i - m_t^i}{m_{t+1}^i} = h\left(\frac{m_{t+1}^i}{z m_t^i}\right),\ i = h, f. \tag{16}$$

A stationary equilibrium satisfies $(y^i - m^i)/m^i = h(1/z)$, $i = h,f$. Thus, $m^i = y^i/[1 + h(1/z)]$, $i=h,f$. Thus,

$$\frac{m^f}{m^h} = \frac{e_t M_t^{sf}}{M_t^{sh}} \Rightarrow e_t = \frac{M_t^{sh}}{M_t^{sh}} \frac{y^f}{y^h}. \tag{17}$$

Now suppose that a consumer holds both home and foreign currencies. Let M_t^{ji} be the time-*i* country *i* demand for country *j*. Then the budget constraints of the consumer born at time *t* in country *j* are

$$\begin{aligned} c_t^{yi} &= y^i - M_t^{hi} / P_t - e_t M_t^{fi} / P_t \\ c_{t+1}^{oi} &= M_t^{hi} / P_{t+1} - e_{t+1} M_t^{fi} / P_{t+1} + T_{t+1}^i,\ i = h, f. \end{aligned} \tag{18}$$

In this perfect certainty world, no consumer will hold a currency that is expected to depreciate. Thus, the no arbitrage condition is $e_t = e$. The exchange rate must be a constant. Substituting this into equation (18) yields

$$\begin{aligned} & c_t^{yi} = y^i - S_t^i / P_t \\ & c_{t+1}^{oi} = S_t^i / P_{t+1} + T_{t+1}^i, \\ & \text{where } S_t^i \equiv M_t^{hi} + e M_t^{fi},\ i = h, f. \end{aligned} \tag{19}$$

With a constant exchange rate, consumers no longer care how their portfolios are allocated. The first-order conditions are

$$y^i - S_t^i / P_t = h\left(\frac{P_t}{P_{t+1}}\right)\left(S_t^i / P_{t+1} + T_{t+1}^i\right), \; i = h, f. \tag{20}$$

We can substitute in the value of the transfers, but it is not possible to solve for the demand for the home currency and the demand for the foreign currency. Instead, we can sum the right- and left-hand sides in equations (20), substitute in market clearing and value of the transfers and let $s_t \equiv (M_t^{st} + eM_t^{sf}) / P_t$. This yields $(y^h + y^f - s_t)/s_t = h(1/z)$. For every constant exchange rate, we can solve for an equilibrium price that satisfies this equation. Thus, the exchange rate is not determined

The intuition is that the exchange rate must be constant to ensure that both currencies are valued. But, once it is constant, consumers do not care about how their portfolios are allocated. So, we have only a single market clearing condition that the world demand for money is equal to the world supply. This allows us to determine the real value of the world money supply, but not the exchange rate.

The real-world analogue to this story is not so much the determination of the dollar pound exchange rate but the exchange rate between pennies and dimes in the United States. The government can arbitrarily set any exchange rate that it wants and that would prevail. If it increases the supply of say, dimes, there is no pressure on the dime to depreciate relative to the penny.

Note that adding uncertainty does not change matters. If output, for example, was random across countries we would conjecture that there was a constant exchange rate, rather than imposing this as a consequence of an arbitrage condition. Once it is conjectured that exchange rates are constant then consumers are indifferent as to how their portfolios are allocated and all we can solve for is the price level.

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